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KERNELS OF MAPS BETWEEN CLASSIFYING SPACES

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ABSTRACT

For homomorphisms between groups, one can divide out the kernel to get an injection. Here, we develop a notion of kernels for maps between classifying spaces of compact Lie groups. We show that the kernel is a normal subgroup in a modified sense and prove a generalization of a theorem of Quillen, namely, a map $f\colon BG \longrightarrow BH_p^\wedge$ is injective, iff the induced map in mod-p cohomology is finite. Moreover, for compact connected Lie groups, every map $f\colon BG \longrightarrow BH_p^\wedge$ factors over a quotient of G in a modified sense and this factorisation is an injection.

1. Introduction

For a homomorphism $\rho\colon G \longrightarrow H$ between groups, we know that the kernel $\ker(\rho)$ of ρ is a normal subgroup of G, which gives rise to an exact sequence

$$\ker(\rho) \longrightarrow G \longrightarrow G/\ker(\rho).$$

The induced homomorphism

$$\bar{\rho}\colon G/\ker(\rho) \longrightarrow H$$

is an injection. In this paper we will develop a analoguos concept for maps between classifying spaces.

To investigate the topological situation we pass to the p-adic completion. We also allow a more general target. Let G be a compact connected Lie group. A space X is called **BG-local** if the evaluation induces an equivalence $\mathrm{map}(BG, X) \simeq X$ between the mapping space and X. The space X is called **almost BG-local** if the evaluation induces an equivalence $\mathrm{map}(BG, X)_{\mathrm{const}} \simeq X$ between

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the component of the constant map const: $BG \longrightarrow X$ and X. This is equivalent to the condition that the loop space ΩX is BG-local.

The situation, we are interested in, is the following

$$\text{(S)}\qquad \begin{array}{l} f\colon BG \longrightarrow X_p^\wedge \text{ is a map, where } G \text{ is a compact Lie group, where} \\ X_p^\wedge \text{ is } p\text{-complete and almost } B\mathbb{Z}/p\text{-local and where} \\ H^*(X;\mathbb{F}_p) \text{ is of finite type.} \end{array}$$

When talking about the kernel of a map $f\colon BG \longrightarrow X_p^\wedge$ as in (S), one has to look for elements $g \in G$ of order a power of p, such that $f|_{B\langle g\rangle}$ is homotopic to the constant map. Here, $\langle g\rangle$ denotes the subgroup generated by g. This leads to a definition of the prekernel, due to Ishiguro [7, 8], and of the kernel, which we explain now.

For every compact Lie group G there exists a maximal p-toral subgroup S_pG, unique up to conjugation, and every p-toral subgroup is subconjugated to S_pG [9]. The group S_pG is called the **p-toral Sylow group** of G. If G is finite, S_pG is the usual p-Sylow group.

For a compact Lie group G, we denote by T_G the maximal torus, by NT_G the normalizer of T_G, and by W_G the Weyl group. Then, S_pG is the counterimage of S_pW_G in NT_G. We also denote S_pG by N_pT_G to indicate that S_pG is the p-toral Sylow subgroup of NT_G, too. T_G is the component of the unit of $N_pT_G = S_pG$.

We define a subgroup $S_{p^\infty}G \subset S_pG$ by the commutative diagram

$$\begin{array}{ccccc} T_{p^\infty} & \longrightarrow & S_{p^\infty}G & \longrightarrow & \pi_0(S_pG) \\ \downarrow & & \downarrow & & \downarrow \\ T_G & \longrightarrow & S_pG & \longrightarrow & \pi_0(S_pG)\ , \end{array}$$

where $T_{p^\infty} \subset T_G$ denotes the subgroup generated by all elements of order a power of p. For a map $f\colon BG \longrightarrow X_p^\wedge$, we define the prekernel of f as

$$\operatorname{preker}(f) := \{g \in S_{p^\infty}G\colon f|_{B\langle g\rangle} \simeq \text{const}\},$$

and the kernel of f as

$$\ker(f) := \operatorname{cl}(\operatorname{preker}(f)),$$

where cl( ) denotes the closure in S_pG or G.

1.1 THEOREM: *Let $f: BG \longrightarrow X_p^\wedge$ be a map as in (S).*

(1) preker(f) *is a subgroup of $S_{p^\infty}G$.*

(2) ker(f) *is a p-toral subgroup of S_pG.*

(3) $f|_{\mathrm{Bker}(f)}$ *is nullhomotopic.*

As the proof in the next section shows, the theorem is true without the assumptions that $H^*(X_p^\wedge;\mathbb{F}_p)$ is of finite type.

These results lead to the following definitions: A map $f: BG \longrightarrow X_p^\wedge$ as in (S) is called **injective** if $\ker(f)$ is the trivial group, and, following [4] or [13], **monic** or **finite** if $H^*(BG;\mathbb{F}_p)$ is finitely generated over $H^*(X_p^\wedge;\mathbb{F}_p)$. In [13] it is proved that the kernel of a homomorphism $\rho: G \longrightarrow H$ between two compact Lie groups is finite and of order coprime to p if and only if the induced map $B\rho: BG \longrightarrow BH_p^\wedge$ is monic. The following statement also generalizes a result of Dwyer and Wilkerson [4, Proposition 4.4]

1.2 THEOREM: *Let $f: BG \longrightarrow X_p^\wedge$ be a map as in (S). Then, f is injective, if and only if f is monic.*

Let $\mathcal{O}(G)$ be the orbit category of G, and let $\mathcal{O}_p(G)$ be the full subcategory, whose objects are homogenous spaces G/P, where P is a p-toral subgroup of S_pG. Usually, $\mathcal{O}_p(G)$ is defined to be the full subcategory of the spaces G/P, where P is any p-toral subgroups of G. Our definition is more convenient for our purpose and gives a homotopy equivalent category. This follows because, up to conjugation, every p-toral subgroup is contained in S_pG [9]. For a subgroup $K \subset S_pG$ and a p-toral subgroup $P \subset S_pG$, we define $K_P := K \cap P$.

1.3 LEMMA: *For a p-toral subgroup $\Gamma \subset S_pG$ the following conditions are equivalent:*

(1) *For every $x \in \Gamma$ of order a power of p and for every $g \in G$, we have $gxg^{-1} \in S_pG$ if and only if $gxg^{-1} \in \Gamma$.*

(2) *For every pair $P, P' \subset S_pG$ of p-toral subgroups and for every $g \in G$, such that $gPg^{-1} \subset P'$, we have $g\Gamma_P g^{-1} \subset \Gamma_{P'}$.*

Proof: $S_{p^\infty}\Gamma \subset \Gamma$ is a dense subset and contains only elements of order a power of p. If Γ satisfies the first condition and if $gPg^{-1} \subset P'$, then $g(S_{p^\infty}\Gamma \cap P)g^{-1} \subset S_{p^\infty}\Gamma \cap P'$. Because P' is closed, this is also true for Γ, which is condition (2). Every element of order a power of p generates a finite p-group. Hence, (2) implies (1) obviously. ∎

A p-toral subgroup $\Gamma \subset S_pG$ is called $\mathcal{O}_p(G)$-**normal**, if Γ satisfies one of the conditions of Lemma 1.3.

1.4 PROPOSITION: *Let $f\colon BG \longrightarrow X_p^\wedge$ be a map as in (S). Then,* $\ker(f) \subset S_pG$ *is $\mathcal{O}_p(G)$-normal.*

Proof: In [7] it is shown that for two subgroups $\Gamma, \Gamma' \subset S_pG$, which are conjugated in G, the restriction $f|_{B\Gamma}$ is nullhomotopic if and only if $f|_{B\Gamma'}$ is nullhomotopic. Hence, for every $x \in \ker(f)$, conjugated elements are also contained in $\ker(f)$.

For a map $f\colon BG \longrightarrow X_p^\wedge$ as in (S), the kernel $\ker(f) \subset S_pG$ is not a normal subgroup of G in general. But, for G connected, $\ker(f)$ is the right invariant, which tells us on which part the map $f\colon BG \longrightarrow X_p^\wedge$ is trivial. To avoid discussions and arguments about homotopy colimits, we study $\mathcal{O}_p(G)$-normal subgroups of N_pT_G. This investigation allows one to prove the following theorem:

1.5 THEOREM: *Let G be a compact connected Lie group and let $f\colon BG \longrightarrow X_p^\wedge$ be a map as in (S). Then, there exists a compact Lie group H and a commutative diagram*

$$\begin{array}{ccc} BG & \xrightarrow{f} & X_p^\wedge \\ {\scriptstyle q}\downarrow & & \Vert \\ BH_p^\wedge & \xrightarrow{\bar{f}} & X_p^\wedge \end{array}$$

such that $\bar{f}\colon BH_p^\wedge \longrightarrow X_p^\wedge$ is injective.

Using a weak form of Theorem 1.1 (1), Ishiguro proved a similiar result for simple Lie groups. [7, 8]. We remark that we have to take the completion of BH. Moreover, as the proof shows, the group H in the theorem is not connected in general. This statement says that, at least, we can divide out by the kernel to make the map injective. Moreover, the homotopy fiber of $BG_p^\wedge \xrightarrow{q} BH_p^\wedge$ is closely related to $\ker(f)$. One might think of this homotopy fiber as being the kernel of f and of q as a surjection. In general, the homotopy fiber of q might not be the completion of the classifying space of a compact Lie group.

The paper is organized as follows: In the next section we prove Theorem 1.1, the third section contains a proof of Theorem 1.2, and in the last section $\mathcal{O}_p(G)$-normal subgroups are discussed to prove Theorem 1.5.

Completion is always meant in the sense of [2].

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2. Prekernels and kernels

We start with the following observation:

2.1 LEMMA: *Let $X_p^\wedge$ be a p-complete almost $B\mathbb{Z}/p$-local space. Then, $X_p^\wedge$ is almost BG-local for every compact Lie group G.*

Proof: A p-complete space $X_p^\wedge$ is always $B\mathbb{Z}/p'$-local for any prime $p' \neq p$. If $X_p^\wedge$ is also $B\mathbb{Z}/p$-local, then, by [11, §9], it follows that $X_p^\wedge$ is $B\pi$-local for any locally finite group π. For every compact Lie group G, there exists a mod-p equivalence $B\pi \longrightarrow BG$, where π is locally finite [6]. So, $X_p^\wedge$ is BG-local. Because for a p-complete space $X_p^\wedge$, the loop space $\Omega(X_p^\wedge)$ is also p-complete, the same arguments apply to show that every p-complete almost $B\mathbb{Z}/p$-local space is almost BG-local for every compact Lie group G. ∎

Let π be a finite group. For $x \in \pi$, we define $\nu(x)$ to be the smallest subgroup of π, which is normal in π and contains x. This is welldefined because the intersection of two normal subgroups is also normal.

2.2 LEMMA: *If π is a finite p-group and noncyclic, then, for every $x \in \pi$, $\nu(x) \subset \pi$ is a proper subgroup and $\nu(x) = \langle yxy^{-1}: y \in \pi\rangle$.*

For a set S of elements of π, we denote by $\langle S\rangle$ the subgroup generated by the elements of S.

Proof: The center of π is nontrivial and contains $\mathbb{Z}/p$ as subgroup. Hence, there exists a central extension $\mathbb{Z}/p \longrightarrow \pi \xrightarrow{q} \overline{\pi} := \pi/\mathbb{Z}/p$. If $\overline{\pi} \cong \mathbb{Z}/p^k$ is cyclic, all central group extensions are given by abelian groups. Thus , as a noncyclic group, $\pi \cong \mathbb{Z}/p \oplus \mathbb{Z}/p^k$ and obviously satisfies the statement.

If $\overline{\pi}$ is noncyclic, we can use an induction over the order of π. Let $\overline{x} := q(x)$. Then, $q(\nu(x)) \subset \nu(\overline{x}) \neq \overline{\pi}$ by induction hypothesis. This shows that $\nu(x) \neq \pi$.

To prove the second part of the statement, we observe that the group $\nu'(x) := \langle yxy^{-1}: y \in \pi\rangle$, generated by all the conjugates of x, is normal in π. Hence, $\nu(x) \subset \nu'(x)$. On the other hand, $yxy^{-1} \in \nu(x)$ for all $y \in \pi$, which shows that $\nu'(x) \subset \nu(x)$. ∎

To prove Theorem 1.1 we need the following result, which may be found in [12], or [7].

2.3 LEMMA: *Let $K \longrightarrow G \xrightarrow{q} H$ be an exact sequence of groups, and let X be an almost BK-local space. Then,*

$$q^*\colon \mathrm{map}(BH, X) \longrightarrow \bigcup_{f|_{BK}\simeq \mathrm{const}} \mathrm{map}(BG, X)_f$$

is an equivalence.

2.4 PROPOSITION: *Let $f\colon B\pi \longrightarrow X_p^\wedge$ be a map, where π is a finite p-group and $X_p^\wedge$ a p-complete and almost $B\mathbb{Z}/p$-local space. Let $\{x_1, \ldots, x_r\}$ be a set of generators. If $f|_{B\langle x_i\rangle} \simeq \mathrm{const}$ for all i, then f is homotopically trivial.*

Proof: We pove the statement by an induction over the order of π. If $\pi \cong \mathbb{Z}/p^k$ is cyclic, there is nothing to show because one of the elements must generate π. If π is noncyclic, by Lemma 2.1, there exists an exact sequence

$$\nu(x_1) \longrightarrow \pi \longrightarrow \overline{\pi} := \pi/\nu(x_1)\ .$$

$\nu(x_1)$ is generated by elements of the form yx_1y^{-1}, and $f|_{B\langle yx_1y^{-1}\rangle} \simeq \mathrm{const}$. The order of $\nu(x_1)$ is smaller than the order of π. By induction hypothesis, $f|_{B\nu(x_1)} \simeq \mathrm{const}$. By Lemma 2.1, $X_p^\wedge$ is $B\nu(x_1)$-local, and Lemma 2.3 establishes an equivalence $\mathrm{map}(B\overline{\pi}, X_p^\wedge) \simeq \bigcup_{g|_{B\nu(x_1)}\simeq\mathrm{const}} \mathrm{map}(B\pi, X_p^\wedge)_g$. In particular, f factors over a map $\overline{f}\colon B\overline{\pi} \longrightarrow X_p^\wedge$.

The quotient $\overline{\pi}$ is generated by the elements $\overline{x}_i := q(x_i)$. The exact sequence

$$\nu(x_1) \cap \langle x_i\rangle \longrightarrow \langle x_i\rangle \longrightarrow \langle \overline{x}_i\rangle$$

and another application of Lemma 2.3 show that $\overline{f}|_{B\langle\overline{x}_i\rangle} \simeq \mathrm{const}$. Thus we can again apply the induction hypothesis, which shows that $\overline{f} \simeq \mathrm{const}$. This finishes the proof. ∎

Now, we are prepared to prove Theorem 1.1.

Proof of Theorem 1.1: Let $x, y \in \mathrm{preker}(f)$. We want to show that $xy \in \mathrm{preker}(f)$ or, more generally, that $f|_{B\langle x,y\rangle} \simeq \mathrm{const}$.

$S_{p^\infty}G$ is a locally finite p group. In particuliar, there exists a sequence

$$\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_r \subset \cdots S_{p^\infty}G$$

of finite p groups such that $S_{p^\infty}G = \bigcup_r \Gamma_r$ [5]. Therefore, $\langle x, y\rangle$ is a finite p-group, and, by the last proposition, $f|_{B\langle x,y\rangle} \simeq \mathrm{const}$. This proves (1).

Let $\Gamma'_r := \Gamma_r \cap \mathrm{preker}(f)$. Then, $\mathrm{preker}(f) = \bigcup_r \Gamma'_r$. By Proposition 2.3, $f|_{B\Gamma'_r}$ is nullhomotopic. Because $X_p^\wedge$ is almost $B\mathbb{Z}/p$-local and hence almost $B\Gamma'_r$-local (Lemma 2.1), $\varprojlim{}^1 \pi_1(\mathrm{map}(B\Gamma'_r, X_p^\wedge)_{\mathrm{const}} \cong \varprojlim{}^1 \pi_1(X_p^\wedge)$ vanishes. The Milnor sequence for calculating the homotopy groups of inverse homotopy limits proves that $f|_{\mathrm{Bpreker}(f)}$ is nullhomotopic.

Lemma 2.5 below shows that $\mathrm{Bpreker}(f) \longrightarrow \mathrm{Bker}(f)$ is a mod-p equivalence. For the p-complete space $X_p^\wedge$, the map $[B(\ker(f), X_p^\wedge] \longrightarrow [B(\mathrm{preker}(f), X_p^\wedge]$ between homotopy classes of maps is a bijection [2]. This implies that $f|_{\mathrm{Bker}(f)}$ is nullhomotopic and proves part (3).

$\ker(f)$ is the closure of $\mathrm{preker}(f)$ in S_pG. Thus, the group of the components of $\ker(f)$ is a finite p-group, and $\ker(f)$ is a p-toral group which is part (2). This finishes the proof. ∎

2.5 Lemma: *Let $f\colon BG \longrightarrow X_p^\wedge$ be a map as in (S). Then,the map*

$$B(\mathrm{preker}(f)) \longrightarrow B(\ker(f))$$

is a mod-p equivalence.

Proof: Let $T(f)$ denote the component of the unit of $\ker(f)$, $T_\infty(f)$ the intersection of $T(f)$ and $S_{p^\infty}G$, and let $\pi := \pi_0(\ker(f)$. These groups fit into the commutative diagram

$$\begin{tikzcd}
T_\infty(f) \arrow[r] \arrow[d] & \mathrm{preker}(f) \arrow[r] \arrow[d] & \pi \arrow[d, equal] \\
T(f) \arrow[r] & \ker(f) \arrow[r] & \pi .
\end{tikzcd}$$

Both rows are exact.

As a locally finite abelian p-group, $T_\infty(f) \cong (\mathbb{Z}/p^\infty)^r \times A$, where A is a finite abelian p-group. Because the closure of $T_\infty(f)$ is $T(f)$, A is trivial, and $T(f) \cong (S^1)^r$. So, $BT_\infty(f) \longrightarrow BT(f)$ is a mod-p equivalence. The Serre spectral sequence for mod-p cohomology for the fibrations in the diagram

$$\begin{tikzcd}
BT_\infty(f) \arrow[r] \arrow[d] & \mathrm{Bpreker}(f) \arrow[r] \arrow[d] & B\pi \arrow[d, equal] \\
BT(f) \arrow[r] & \mathrm{Bker}(f) \arrow[r] & B\pi
\end{tikzcd}$$

proves the statement. ∎

3. Injective and monic maps

In this section we proof Theorem 1.2. Let $f\colon BG \longrightarrow X_p^\wedge$ be a map as in (S). Let $\mathcal{A}_p(G)$ denote the Quillen category. The objects are given by elementary abelian p-subgroups and the morphisms by conjugation in G [13]. To get a finite category, we take only one object for every isomorphism class of objects, i.e. for every conjugacy class of a group. The Quillen map

$$\phi\colon H^*(BG;\mathbb{F}_p) \longrightarrow \lim_{\substack{\longleftarrow \\ V\in\mathcal{A}_p(G)}} H^*(BV;\mathbb{F}_p)$$

is $H^*(BG;\mathbb{F}_p)$-linear and an F-isomorphism; i.e. kernel and cokernel are nilpotent [13, Theorem 7.2]. Let

$$B := \operatorname{im}(\phi\circ f^*) \subset \lim_{\substack{\longleftarrow \\ V\in\mathcal{A}_p(G)}} H^*(BV;\mathbb{F}_p) \subset \prod_{V\in\mathcal{A}_p(G)} H^*(BV;\mathbb{F}_p)$$

be the image of $H^*(X_p^\wedge;\mathbb{F}_p)$.

Proof of Theorem 1.2: First, we assume that f is injective and show that f is monic. For any elementary abelian p-subgroup $V\subset G$, the restriction $f|_{BV}$ is also injective. By [4, Proposition 4.4], which is the analogous statement of Theorem 1.2 for elementary abelian p-groups, this implies that $H^*(BV;\mathbb{F}_p)$ is a finitely generated module over $H^*(X_p^\wedge;\mathbb{F}_p)$ and over B. Because $\mathcal{A}_p(G)$ is a finite category, $\prod_{V\in\mathcal{A}_p(G)} H^*(BV;\mathbb{F}_p)$ is a finitely generated module over B and a finitely generated algebra over $\mathbb{F}_p$. Therefore, B is also a finitely generated algebra over $\mathbb{F}_p$ [1, Proposition 7.8] and hence noetherian. This implies that $\lim_{\substack{\longleftarrow \\ V\in\mathcal{A}_p(G)}} H^*(BV;\mathbb{F}_p)$, as a submodule of $\prod_{V\in\mathcal{A}_p(G)} H^*(BV;\mathbb{F}_p)$, is a finitely generated module over B and over $H^*(X_p^\wedge;\mathbb{F}_p)$.

If f is not monic, i.e the finitely generated $\mathbb{F}_p$-algebra $H^*(BG;\mathbb{F}_p)$ is not a finitely generated module over $H^*(X_p^\wedge;\mathbb{F}_p)$, there exists an element $y\in H^*(BG;\mathbb{F}_p)$ such that $\{y^i\colon i\in\mathbb{N}\}$ is a set of linearly independent elements over $H^*(X;\mathbb{F}_p)$. By the above considerations, for $r\in\mathbb{N}$ big enough, there exists a relation $\phi(y^r)=\sum_{i=0}^{r-1} x_i\phi(y^i)$ with $x_i\in H^*(X_p^\wedge;\mathbb{F}_p)$. That is that $y^r-\sum_{i=0}^{r-1}x_iy^i$ is in the kernel of ϕ and hence nilpotent. Thus, for $s\in\mathbb{N}$ big enough,

$$0=\Big(y^r-\sum_{i=0}^{r-1}x_iy^i\Big)^s = y^{rs}-\sum_{j=0}^{r-1}x_j'y^j$$

for suitable $x_i' \in H^*(X_p^\wedge;\mathbb{F}_p)$. This is a contradiction and proves that f is monic.

Now, we assume that f is monic. Let $\mathbb{Z}/p \subset G$ be a subgroup of G. Up to conjugation, $\mathbb{Z}/p$ is contained in S_pG. By [13], $H^*(B\mathbb{Z}/p;\mathbb{F}_p)$ is finitley generated over $H^*(BG;\mathbb{F}_p)$, and therefore, also over $H^*(X_p^\wedge;\mathbb{F}_p)$. That is that the map $B\mathbb{Z}/p \longrightarrow BG \xrightarrow{f} X_p^\wedge$ is homotopically nontrivial. This implies that $\ker(f) = \{1\}$ and that f is injective. ∎

4. $\mathcal{O}_p(G)$-normal subgroups

For every compact connected Lie group G, there exists a finite covering

$$K \longrightarrow \widetilde{G} \xrightarrow{\ \alpha\ } G\,,$$

where $\widetilde{G} \cong G_s \times T$ is a product of a simply connected Lie group G_s and a torus T. $G_s \cong \prod G_i$ is a product of simply connected simple Lie groups. K is a finite central subgroup of $\widetilde{G}$. The group $\widetilde{G}$ is unique up to isomorphisms. Such coverings we call universal finite.

4.1 Lemma: *Let $K \to \widetilde{G} \xrightarrow{\alpha} G$ be an exact sequence of compact Lie groups, K finite and $\widetilde{G}$ and G connected. Let $f\colon BG \to X_p^\wedge$ be a map as in (S).*

(1) *The sequence*

$$S_pK \to \operatorname{preker}(f \circ B\alpha) \to \operatorname{preker}(f)$$

is exact.

(2) $\ker(f \circ B\alpha) \to \ker(f)$ *is an epimorphism, and* $\ker(f\circ B\alpha) = S_p\alpha^{-1}\ker(f)$.

Proof: Obviously, $\alpha^{-1}(\operatorname{preker}(f)) \cap S_{p^\infty}\widetilde{G}$ is contained in $\operatorname{preker}(f\circ B\alpha)$. In particuliar, $S_pK \subset \operatorname{preker}(f\circ B\alpha)$. Let $\Gamma := \operatorname{preker}(f\circ B\alpha)/S_pK \supset \operatorname{preker}(f)$. Then, by Lemma 2.3,

$$\operatorname{map}(B\Gamma, X_p^\wedge) \simeq \coprod_{g|_{BS_pK}\simeq \mathrm{const}} \operatorname{map}(B\operatorname{preker}(f\circ B\alpha), X_p^\wedge)_g.$$

This implies that $f|_{B\Gamma}$ is homotopically trivial, and hence, that $\Gamma \subset \operatorname{preker}(f)$, which establishes the desired sequence of (1).

To prove the second statement we first observe that in this case epimorphism are maintained under taking closures. The second part in (2) follows from the facts that, as a p-toral group, $\ker(f\circ B\alpha) \subset S_p\alpha^{-1}\ker(f)$, and that $f|_{BS_p\alpha^{-1}\ker(f)}$ is homotopically trivial. ∎

4.2 LEMMA: *Let $K \to \widetilde{G} \xrightarrow{\alpha} G$ be an exact sequence of compact Lie groups, K finite and $\widetilde{G}$ and G connected. Let $P \subset S_pG$ be a p-toral subgroup. Then, P is $\mathcal{O}_p(G)$-normal if and only if $S_p\alpha^{-1}(P) \subset S_p\widetilde{G}$ is $\mathcal{O}_p(\widetilde{G})$-normal.*

Proof: Let $\widetilde{Q} := \alpha^{-1}(P)$ and $\widetilde{P} := S_p\widetilde{Q}$. The composition $\widetilde{P} \to \widetilde{Q} \to P$ is an epimorphism. This follows, because $\widetilde{P}$ and $\widetilde{Q}$ have identical components of the unit, and because, passing to the components, the composition $\pi_0(\widetilde{P}) = S_p\pi_0(\widetilde{Q}) \to \pi_0(P)$ is an epimorphism of finite groups.

$K \subset \widetilde{G}$ is a central subgroup. The multiplication μ: $(K \cap \widetilde{Q}) \times \widetilde{P} \to \widetilde{Q}$ fits into the pull back diagram

$$\begin{tikzcd}
(K \cap \widetilde{Q}) \times \widetilde{P} \arrow[r, "\mu"] \arrow[d] & \widetilde{Q} \arrow[d, "\alpha"] \\
\widetilde{P} \arrow[r, "\alpha"] & P .
\end{tikzcd}$$

Thus, μ is an epimorphism, and $\widetilde{P} \subset \widetilde{Q}$ is a normal subgroup. That is that $\widetilde{P}$ is the only p-toral Sylow subgroup of $\widetilde{Q}$.

Every element $x \in P$ of order a power of p has a lift $\tilde{x} \in \widetilde{P}$, also of order a power of p. Let $\tilde{g} \in \widetilde{G}$ and $g := \alpha(\tilde{g})$. Then, $\tilde{g}\tilde{x}\tilde{g}^{-1} \in S_p\widetilde{G}$ if and only if $gxg^{-1} \in S_pG$, and, because $\widetilde{P} \subset \widetilde{Q}$ is the only p-toral Sylow subgroup, $\tilde{g}\tilde{x}\tilde{g}^{-1} \in \widetilde{P}$ if and only if $gxg^{-1} \in P$. This proves the statement. ∎

Lemma 4.1 and Lemma 4.2 reduce the calculation of kernels and $\mathcal{O}_p(G)$-normal subgroups, G connected, to the case of products of simply connected Lie groups and tori. Let $\widetilde{G} \cong G_s \times T$ be such a product. In order to describe $\mathcal{O}_p(\widetilde{G})$-normal subgroups, we associate for every prime p a subgroup $H(G,p)$ to each simply connected simple Lie group G. We define

$$H(G,p) := \begin{cases} NT_G & \text{if } (p, |W_G|) = 1, \\ SU(2) \rtimes \mathbb{Z}/2 & \text{if } G = G_2 \text{ and } p = 3, \\ G & \text{else.} \end{cases}$$

We define $H(G_s,p) := \prod H(G_i,p)$ for a product $G_s = \prod G_i$ of simply connected simple Lie groups. Then, $BH(G_s,p) \hookrightarrow BG$ is a mod-p equivalence. If $p = 3$ and $G = G_2$, this follows from the isomorphism

$$H^*(BG_2;\mathbb{Z}/p) \cong H^*(BSU(2);\mathbb{Z}/p)^{\mathbb{Z}/2},$$

and if $(p, |W_{G_s}|) = 1$ from the isomorphism $H^*(BG_s;\mathbb{Z}/p) \cong H^*(BT_{G_s};\mathbb{Z}/p)^{W_{G_s}}$.

4.3 PROPOSITION: *Let $\widetilde{G} = \prod G_i \times T$ be a product of simply connected simple Lie groups G_i and a torus T. Let $\Gamma \subset N_pT_{\widetilde{G}}$ be a $\mathcal{O}_p(\widetilde{G})$-normal subgroup. Then, we can split $G_s = G' \times G''$ such that $\Gamma \cong N_pT_{G'} \times \hat{\Gamma}$ and $\hat{\Gamma} \subset T_{G''} \times T$. Moreover, $\hat{\Gamma}$ is normal in $H(G'',p) \times T$ and the image of $\hat{\Gamma}$ in G'' is finite.*

We postpone the proof. This result enables us to prove Theorem 1.5.

Proof of Theorem 1.5: Let $f\colon BG \longrightarrow X_p^\wedge$ be a map as in (S), and let

$$K \longrightarrow \widetilde{G} \longrightarrow G$$

be a universal finite covering, where $\widetilde{G} \cong G_s \times T$. By the last proposition and Proposition 1.3, $G_s \cong G' \times G''$ and $\ker(f \circ B\alpha) \cong N_pT_{G'} \times \hat{\Gamma}$. Now, we define $H = (H(G'',p) \times T)/\hat{\Gamma}$. The classical kernel of the projection $G' \times H(G'',p) \times T \longrightarrow H$ is given by $G' \times \hat{\Gamma}$, which contains K by construction. We get a commutative diagram

$$\begin{tikzcd}
B(G' \times H(G'',p) \times T) \arrow[r, "Bi"] \arrow[d, "B\hat{\alpha}"'] & B\widetilde{G} \arrow[r, "f \circ B\alpha"] \arrow[d, "B\alpha"'] & X_p^\wedge \arrow[d, equal] \\
B(G' \times H(G'',p) \times T)/K \arrow[r, "B\bar{i}"] \arrow[d, "\hat{q}"'] & BG \arrow[r, "f"] \arrow[d, "q"'] & X_p^\wedge \arrow[d, equal] \\
BH_p^\wedge \arrow[r, equal] & BH_p^\wedge \arrow[r, "\bar{f}"] & X_p^\wedge .
\end{tikzcd}$$

Bi and $B\bar{i}$ are mod-p equivalences. $BH_p^\wedge$ is p-complete, because $\pi_1(BH)$ is a finite group [2]. This establishes the map $q\colon BG \longrightarrow BH_p^\wedge$. The quotient $(G' \times \hat{\Gamma})/K$ is a normal subgroup of $(G' \times H(G'',p) \times T)/K$, and $X_p^\wedge$ is almost $B(G' \times \hat{\Gamma})/K$-local (Lemma 2.1). Therefore, the map $f \circ B\bar{i}$ factors over $\bar{f}\colon BH_p^\wedge \longrightarrow X_p^\wedge$ (Lemma 2.3). Moreover, $\bar{f} \circ q \simeq f$, because $\bar{f} \circ \hat{q} \simeq f \circ B\bar{i}$. This proves the first half of Theorem 1.5.

By Lemma 4.1 , $\ker(f \circ B\alpha \circ Bi) \xrightarrow{\beta \circ \hat{\alpha}} \ker(\bar{f})$ is an epimorphism. This shows that $\bar{f}$ is injective. ∎

In the rest of this section, we prove Proposition 4.3.

Proof of Proposition 4.3: The subgroup $\Gamma_T = \Gamma \cap T_{\widetilde{G}}$ is invariant under the Weyl group action. In particuliar, $W_{\widetilde{G}}$ acts on the component Γ_e of the unit of Γ, and $H^2(B\Gamma_e; \mathbb{Q})$ is a $W_{\widetilde{G}}$-submodule of

$$H^2(BT_{\widetilde{G}}; \mathbb{Q}) \cong \bigoplus H^2(BG_i; \mathbb{Q}) \oplus H^2(BT; \mathbb{Q}) .$$

The first summands are irreducible. Thus, $\Gamma_T \cap T_{G_i} = T_{G_i}$ or the intersection is trivial.

Let G' be the product of all factors G_i with $T_{G_i} \subset \Gamma$, and G'' the product of the other factors of $\widetilde{G}$. Let $x \in N_pT_{G'}$ but $x \notin T_{G'}$. Then x is conjugated to an element in $T_{G'}$ and therefore, $x \in \Gamma$. This implies that $N_pT_{G'} \subset \Gamma$. Moreover, $N_pT_{G'}$ is a normal subgroup of Γ. In the commutative diagram

$$\begin{tikzcd} N_pT_{G'} \arrow[r, "i"] \arrow[d, equal] & \Gamma \arrow[r] \arrow[d] & \hat{\Gamma} := \Gamma/N_pT_{G'} \arrow[d] \\ N_pT_{G'} \arrow[r, "j"] & N_pT_{G'} \times N_pT_{G''} \times T \arrow[r] & N_pT_{G''} \times T \end{tikzcd}$$

both rows are central extensions. The map j has a section, which establishes also a section for i. Therefore, the upper sequence is the trivial extension, $\Gamma \cong N_pT_{G'} \times \hat{\Gamma}$, and $\hat{\Gamma} \subset N_pT_{G''} \times T$. This is the first part of the statement.

Because $\Gamma_e \subset N_pT_{G'} \times T$, the image of $\hat{\Gamma}$ in G'' is a finite group and $\mathcal{O}_p(G'')$-normal. We have to investigate finite $\mathcal{O}_p(G)$-normal subgroups of simply connected simple Lie groups. The following proposition finishes the proof. ∎

4.4 PROPOSITION: *Let G be a simply connected simple Lie group, and $\Gamma \subset N_pT_G$ a $\mathcal{O}_p(G)$-normal finite p-subgroup.*

(1) *If p divides $|W_G|$, and if $G \neq G_2$ or $p \neq 3$, then Γ is central in G.*
(2) *If $G = G_2$ and $p = 3$, then Γ is central in $SU(3)$ and hence normal in $H(G_2, 3)$.*
(3) *If $(p, |W_G|) = 1$, then Γ is normal in $H(G,p)$.*

Proof: If $x \in \Gamma$ then $txt^{-1}x^{-1} \in \Gamma$ for all $t \in T_G$. Because Γ is finite, all the commutators are trivial. Thus, x centralizes T_G. Because $C_G(T_G) = T_G$, $\Gamma = \Gamma_T \subset T_G$.

If $(p, |W_G|) = 1$, Γ is normal in $NT_G = H(G,p)$. If p divides $|W_G|$, we have to prove that Γ is central in G or, for $G = G_2$ and $p = 3$, central in $SU(3)$. Let $x \in \Gamma \setminus Z(G)$ and $x^{p^k} = 1$. Using this element, we construct an element $\Gamma \setminus T_G$, which gives a contradiction. This is done by a case by case checking and very much in the flavour of the proof of [7, Theorem 2'].

Before we start, we make two observations. First, if (1) is true for two groups G and H, then it is obviously true for the product $G \times H$ and, Second, if $G \xrightarrow{q} \overline{G}$ is a covering of connected Lie groups, then (1) is true for G if and only if $\overline{G}$ satisfies condition (1). To see this, we consider a finite $\mathcal{O}_p(G)$-normal p-subgroup

$\Gamma \subset N_pT_G$. Then, $\Gamma' := \langle \Gamma, S_pZ(G)\rangle$ is also a finite $\mathcal{O}_p(G)$-normal p-subgroup. Because $\Gamma' = S_p(q^{-1}(q(\Gamma')))$, the group $q(\Gamma')$ is $\mathcal{O}_p(\overline{G})$-normal if and only if Γ' is $\mathcal{O}_p(G)$-normal (Lemma 4.2), and $q(\gamma') \subset \overline{G}$ is central if and only if $\Gamma' \subset G$ is central.

Let $G = SU(n)$, $n \geq p$. Up to conjugation, x can be represented by a diagonal matrix $D = D(a_1, \dots, a_n)$, where a_i is a p^k-th root of unity, and $a_1 \neq a_2$. The element $y = D(a_2, a_1, a_3, \dots, a_n)$ is conjugate to x and thus, $xy^{-1} = D(a_1a_2^{-1}, a_1^{-1}a_2, 1, \dots, 1) \in \Gamma$. Because $a_1a_2^{-1} \neq 1$, conjugates of xy^{-1} generate a subgroup of T_G, which contains the maximal elementary abelian p-subgroup V_G of T_G. Every element of order p is conjugate to an element in V_G. This gives a contradiction. For $G = U(n)$, $n \geq p$, the same proof works.

Let $G = Sp(n)$, $n \geq p$. Then, $U(n)$ and $SU(2)^n$ are subgroups of maximal rank. Therefore, $\Gamma \subset T_{Sp(n)}$ is central in $U(n)$ and $SU(2)^n$. But $Z(U(n)) \cap Z(SU(2)^n) = \mathbb{Z}/2 = Z(Sp(n))$.

By the above observation, the case of $G = Spin(n)$, $n \geq 2p$ and $n \geq 5$, can be reduced to the case of $SO(n)$. Let $n = 2k$ or $n = 2k+1$. Then, $U(k) \subset SO(2k) \subset SO(2k+1)$ is a subgroup of maximal rank, and Γ is central in $U(k)$. For $n \geq 5$, the only $W_{SO(2k)}$ or $W_{SO(2k+1)}$-invariant subgroup of $S^1 = Z(U(k))$ is $\mathbb{Z}/2$. For $SO(2k)$, $\mathbb{Z}/2 \subset U(k) \subset SO(2k)$ is central, and for $SO(2k+1)$, $\mathbb{Z}/2 \subset U(k) \subset SO(2k+1)$ is conjugate to a subgroup in $NT_{SO(2k+1)} \setminus T_{SO(2k+1)}$, which proves the statement for $SO(n)$.

Let G be an exceptional Lie group and p a divisor of $|W_G|$. In this case, we choose subgroups of maximal rank, given by the following list:

$G_2:$	$p = 2,3$	$H = SU(3)$		
$F_4:$	$p = 2$	$H = SU(3) \times_{\mathbb{Z}/3} SU(3)$	$p = 3$	Spin(9)
$E_6:$	$p = 2,5$	$H = SU(3)^3/\mathbb{Z}/3$	$p = 3$	$H = SU(2) \times_{\mathbb{Z}/2} SU(6)$
$E_7:$	$p = 3,5,7$	$H = SU(8)/\mathbb{Z}/2$	$p = 2$	$H = SU(3) \times_{\mathbb{Z}/3} SU(6)$
$E_8:$	$p = 3,5,7$	$H = SU(2) \times_{\mathbb{Z}/2} E_7$	$p = 2$	$H = SU(9)/\mathbb{Z}/3$.

Beside the case $G = G_2$ and $p = 3$, the inclusion $H \subset G$ always induces an isomorphism $S_pZ(H) \cong S_pZ(G)$ between the p-Sylow subgroups of the centers. The data may be obtained from [9], where one can find a complete list of maximal subgroups of maximal rank of the exeptional Lie groups, and from [14].

Now, we can argue as follows: Let $\Gamma \subset N_pT_G$ be an $\mathcal{O}_p(G)$-normal subgroup. Then, by induction over the rank, by the above observations, and by the already calculated cases, Γ is central in H and hence, central in G.

For $G = G_2$ and $p = 3$, the argument only shows that Γ is central in $SU(3)$. That is $\Gamma = \mathbb{Z}/3$ or Γ is the trivial group. In both cases, Γ is normal in $SU(3) \rtimes \mathbb{Z}/2 = H(G_2, 3)$. This finishes the last open case and the proof of the proposition. ∎

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